

Chemically Self-Consistent Modeling of the Globular Cluster NGC 2808 and its Effects on the Inferred Helium Abundance of Multiple Stellar Populations

2 EMILY M. BOUDREAU¹, BRIAN C. CHABOYER¹, AMANDA ASH², RENATA EDAES HOH¹, AND GREGORY FEIDEN²

3 ¹*Department of Physics and Astronomy, Dartmouth College, Hanover, NH 03755, USA*

4 ²*Department of Physics and Astronomy, University of North Georgia, Dahlonega, GA 30533, USA*

5 ABSTRACT

6 The helium abundances in the multiple populations that are now known to comprise all closely
7 studied Milky Way globular clusters are often inferred by fitting isochrones generated from stellar
8 evolutionary models to globular cluster photometry. It is therefore important to build stellar models
9 that are chemically self-consistent in terms of their structure, atmosphere, and opacity. In this work
10 we present the first chemically self-consistent stellar models of the Milky Way globular cluster NGC
11 2808 using MARCS model atmospheres, OPLIB high-temperature radiative opacities, and AESOPUS
12 low-temperature radiative opacities. These stellar models were fit to the NGC 2808 photometry using
13 *Fidanka*, a new software tool that was developed to optimally fit cluster photometry to isochrones
14 and for population synthesis. *Fidanka* can determine, in a relatively unbiased way, the ideal number of
15 distinct populations that exist within a dataset and then fit isochrones to each population. We achieve
16 this outcome through a combination of Bayesian Gaussian Mixture Modeling and a novel number
17 density estimation algorithm. Using *Fidanka* and F275W-F814W photometry from the Hubble UV
18 Globular Cluster Survey we find that the helium abundance of the second generation of stars in NGC
19 2808 is higher than the first generation by $15 \pm 3\%$. This is in agreement with previous studies of
20 NGC 2808. **This work, along with previous work by Dotter et al. (2015) focused on NGC**
21 **6752, demonstrates that chemically self-consistent models of globular clusters do not**
22 **significantly alter inferred helium abundances, and are therefore unlikely to be worth the**
23 **significant additional time investment.**

24 *Keywords:* Globular Clusters (656), Stellar evolutionary models (2046)

25 1. INTRODUCTION

26 Globular clusters (GCs) are among the oldest observ-
27 able objects in the universe (Peng et al. 2011). They
28 are characterized by high densities with typical half-
29 light radii of ≤ 10 pc (van den Bergh 2010), and typi-
30 cal masses ranging from 10^4 – $10^5 M_{\odot}$ (Brodie & Strader
31 2006) — though some GCs are significantly larger than
32 these typical values (e.g. ω Cen, Richer et al. 1991).
33 GCs provide a unique way to probe stellar evolution
34 (Kalirai & Richer 2010), galaxy formation models
35 (Boylan-Kolchin 2018; Kravtsov & Gnedin 2005), and
36 dark matter halo structure (Hudson & Robison 2018).

37 The traditional view of globular clusters is that they
38 consist of a single stellar population (SSP, in some publi-
39 cations this is referred to as a Simple Stellar Population).
40 This view was supported by spectroscopically uniform
41 heavy element abundances (Carretta et al. 2010; Bastian
42 & Lardo 2018) across most clusters (M54 and ω Cen are
43 notable exceptions, see Marino et al. (2015) for further
44 details), and the lack of evidence for multiple stellar pop-
45 ulations (MPs) in past color-magnitude diagrams of GCs
46 (i.e. Sandage 1953; Alcaino 1975). However, over the
47 last 40 years non-trivial star-to-star light-element abun-
48 dance variations have been observed (i.e. Smith 1987)
49 and, in the last two decades, it has been definitively
50 shown that most, if not all, Milky Way GCs have MPs
51 (Gratton et al. 2004, 2012; Piotto et al. 2015). The lack
52 of photometric evidence for MPs prior to the 2000, can
53 be attributed to the more narrow color bands available,

54 until very recently, to ground-based photometric surveys
55 (Milone et al. 2017).

56 The prevalence of multiple populations in GCs is so
57 distinct that the proposed definitions for what consti-
58 tutes a globular cluster now often center on the existence
59 of MPs (e.g. Carretta et al. 2010). Whereas people have
60 have often tried to categorize objects as GCs through
61 relations between half-light radius, density, and surface
62 brightness profile, in fact many objects which are gener-
63 ally thought of as GCs don't cleanly fit into these cuts
64 (Peebles & Dicke 1968; Brown et al. 1991, 1995; Bekki
65 & Chiba 2002). Consequently, Carretta et al. (2010)
66 proposed a definition of GCs based on observed chem-
67 ical inhomogeneities in their stellar populations. The
68 modern understanding of GCs then is not simply that
69 of a dense cluster of stars that may have chemical inho-
70 mogeneities and multiple populations; rather, it is one
71 where those chemical inhomogeneities and multiple pop-
72 ulations themselves are the defining elements of a GC.

73 All Milky Way globular clusters studied in detail show
74 populations enriched in He, N, and Na while also be-
75 ing depleted in O and C (Piotto et al. 2015; Bastian &
76 Lardo 2018). **Further, studies of Magellanic Cloud**
77 **massive clusters have shown that these light el-**
78 **ement abundance variations exist in clusters as**
79 **young as ~ 2 Gyr but not in younger clusters**
80 **(Martocchia et al. 2019) while there is also evi-**
81 **dence of nitrogen variability in the ~ 1.5 Gyr old**
82 **cluster NGC 1783 (Cadelano et al. 2022).** These
83 light element abundance patterns also are not strongly
84 correlated with variations in heavy element abundance,
85 resulting in spectroscopically uniform Fe abundances be-
86 tween populations (**though recent work indicates**
87 **that there may be [Fe/H] variations within the**
88 **first population, e.g. Legnardi et al. 2022; Lardo**
89 **et al. 2022). Further, high-resolution spectral studies**
90 reveal anti-correlations between N-C abundances, Na-O
91 abundances, and potentially Al-Mg (Snedden et al. 1992;
92 Gratton et al. 2012). Typical stellar fusion reactions
93 can deplete core oxygen; however, the observed abun-
94 dances of Na, Al, and Mg cannot be explained by the
95 CNO cycle (Prantzos et al. 2007). Consequently, glob-
96 ular cluster populations must be formed by some novel
97 means.

98 Formation channels for these multiple populations re-
99 main a point of debate among astronomers. Most pro-
100 posed formation channels consist of some older, more
101 massive population of stars polluting the pristine clus-
102 ter media before a second population forms, now en-
103 riched in heavier elements which they themselves could
104 not have generated (for a detailed review see Gratton
105 et al. 2012). The four primary candidates for these pol-

106 luters are asymptotic giant branch stars (AGBs, Ventura
107 et al. 2001; D'Ercole et al. 2010), fast rotating mas-
108 sive stars (FRMSs, Decressin et al. 2007), super mas-
109 sive stars (SMSs, Denissenkov & Hartwick 2014), and
110 massive interacting binaries (MIBs, de Mink et al. 2009;
111 Bastian & Lardo 2018).

112 Hot hydrogen burning (i.e. proton capture), material
113 transport to the surface, and material ejection into the
114 intra-cluster media are features of each of these models
115 and consequently they can all be made to *qualitatively*
116 agree with the observed elemental abundances. How-
117 ever, none of the standard models can currently account
118 for all specific abundances (Gratton et al. 2012). AGB
119 and FRMS models are the most promising; however,
120 both models have difficulty reproducing severe O deple-
121 tion (Ventura & D'Antona 2009; Decressin et al. 2007).
122 Moreover, AGB and FRMS models require significant
123 mass loss ($\sim 90\%$) between cluster formation and the
124 current epoch — implying that a significant fraction of
125 halo stars formed in GCs (Renzini 2008; D'Ercole et al.
126 2008; Bastian & Lardo 2015).

127 In addition to the light-element anti-correlations ob-
128 served, it is also known that second populations are sig-
129 nificantly enhanced in helium (Piotto et al. 2007, 2015;
130 Latour et al. 2019). Depending on the cluster, helium
131 mass fractions as high as $Y = 0.4$ have been inferred (e.g.
132 Milone et al. 2015a). However, due to both the relatively
133 high and tight temperature range of partial ionization
134 for He and the efficiency of gravitational settling in core
135 helium burning stars, the initial He abundance of glob-
136 ular cluster stars cannot be observed; consequently, the
137 evidence for enhanced He in GCs originates from com-
138 parison of theoretical stellar isochrones to the observed
139 color-magnitude-diagrams of globular clusters. There-
140 fore, a careful handling of chemistry is essential when
141 modeling with the aim of discriminating between MPs;
142 yet only a very limited number of GCs have been stud-
143 ied with chemically self-consistent (structure and atmo-
144 sphere) isochrones (e.g. Dotter et al. 2015, NGC 6752).

145 NGC 2808 is the prototype globular cluster to host
146 multiple populations. Various studies since 2007 have
147 identified that it may host anywhere from two to five
148 stellar populations. These populations have been iden-
149 tified both spectroscopically (i.e. Carretta et al. 2004;
150 Carretta 2006; Carretta et al. 2010; Gratton et al. 2011;
151 Carretta 2015; Hong et al. 2021) and photometrically
152 (i.e. Piotto et al. 2007, 2015; Milone et al. 2015a, 2017;
153 Pasquato & Milone 2019). Note that recent work (Valle
154 et al. 2022) calls into question the statistical significance
155 of the detections of more than two populations in the
156 spectroscopic data. Here we present **the first stel-**

lar structure and evolutionary models built in a chemically self-consistent manner of NGC 2808.

We model the photometry of the primordial population (hereafter P1) and the helium enriched population (hereafter P2). Milone et al. (2015a) identifies five populations within NGC 2808, given that the aim of this work is not to identify sub-populations; rather, to measure the effect that chemical self consistent stellar structure and evolutionary have on the inferred helium abundance for the two most extreme cases, we do not consider more than those two populations. We use archival photometry from the Hubble UV Globular Cluster Survey (HUGS) (Piotto et al. 2015; Milone et al. 2017) in the F275W and F814W passbands to characterize multiple populations in NGC 2808 (Milone et al. 2015a,b) (This data is available on MAST, Piotto 2018). Additionally, we present a likelihood analysis of the photometric data of NGC 2808 to determine the number of populations present in the cluster.

2. CHEMICAL CONSISTENCY

There are three primary areas in which the stellar models must be made chemically consistent: the atmospheric boundary conditions, the opacities, and interior abundances. The interior abundances are relatively easily handled by adjusting parameters within our stellar evolutionary code. However, the other two areas are more difficult to make consistent. Atmospheric boundary conditions and opacities must both be calculated with a consistent set of chemical abundances outside of the stellar evolution code. **Nearly all prior efforts at modeling multiple stellar populations in globular clusters have adjusted the abundances used in the atmospheric interior models, and in the high temperature opacities, but have not self-consistently modified the corresponding low-temperature opacities and surface boundary conditions, as these are found from stellar atmosphere codes, and not the stellar interior codes which are used to create stellar models and isochrones. In this work, as in Dotter (2016), the stellar interior models are chemically self-consistent with the stellar atmosphere models.** For evolution, we use the Dartmouth Stellar Evolution Program (DSEP) (Dotter et al. 2008), a well-tested 1D stellar evolution code which has a particular focus on modeling low mass stars ($\leq 2 M_{\odot}$)

2.1. Atmospheric Boundary Conditions

Certain assumptions, primarily that the radiation field is at equilibrium and radiative transport is dif-

fusive (Salaris & Cassisi 2005), made in stellar structure codes, such as DSEP, are valid when the optical depth of a star is large. However, in the atmospheres of stars, the number density of particles drops low enough and the optical depth consequently becomes small enough that these assumptions break down, and separate, more physically motivated, plasma- modeling code is required. Generally, structure code will use tabulated atmospheric boundary conditions generated by these specialized codes, such as ATLAS9 (Kurucz 1993), PHOENIX (Husser et al. 2013), MARCS (Gustafsson et al. 2008), and MPS-ATLAS (Kostogryz et al. 2023). Often, because the boundary conditions are expensive to compute, they are not updated as interior abundances vary.

One key element when building chemically self-consistent models of NGC 2808 is the incorporation of new atmospheric boundary conditions with the same elemental abundances as the structure code. We use atmospheres generated from the MARCS grid of model atmospheres (Plez 2008). MARCS provides one-dimensional, hydrostatic, plane-parallel and spherical LTE atmospheric models (Gustafsson et al. 2008). Model atmospheres are made to match the spectroscopically measured elemental abundances of Milone et al. (2015a) populations A&E. Moreover, for each population, atmospheres with various helium mass fractions are generated. These range from $Y=0.24$ to $Y=0.36$ in steps of 0.03. All atmospheric models are computed to an optical depth of $\tau = 100$ where their temperature and pressure serve as boundary conditions for the structure code. In general, enhancing helium in the atmosphere has only a small impact on the atmospheric temperature profile, while leading to a drop in the pressure by $\sim 10 - 20\%$.

2.2. Opacities

In addition to the atmospheric boundary conditions, both the high and low temperature opacities used by DSEP must be made chemically consistent. Here we use OPLIB high temperature opacity tables (Colgan et al. 2016) retrieved using the TOPS web-interface. Retrieval of high temperature opacities is done using pyTOPSScrape, first introduced in Boudreaux & Chaboyer (2023). Low temperature opacity tables are retrieved from the Aesopus 2.0 web-interface (Marigo & Aringer 2009; Marigo et al. 2022). Ideally, these opacities would be the same used in the atmospheric models. However, the opacities used in the MARCS models are not publicly available. As such, we use the opacities provided by the TOPS and Aesopus 2.0 web-interfaces.

3. STELLAR MODELS

Table 1. Population Composition

Element	P1 (A)	P2 (E)	Element	P1 (A)	P2 (E)
Li	-0.08	—	In	-1.46	—
Be	0.25	—	Sn	-0.22	—
B	1.57	—	Sb	-1.25	—
C	6.87	5.91	Te	-0.08	—
N	6.42	6.69	I	-0.71	—
O	7.87	6.91	Xe	-0.02	—
F	3.43	—	Cs	-1.18	—
Ne	7.12	6.7	Ba	1.05	—
Na	5.11	5.7	La	-0.03	—
Mg	6.86	6.42	Ce	0.45	—
Al	5.21	6.61	Pr	-1.54	—
Si	6.65	6.77	Nd	0.29	—
P	4.28	—	Pm	-99.0	—
S	6.31	5.89	Sm	-1.3	—
Cl	-1.13	4.37	Eu	-0.61	—
Ar	5.59	5.17	Gd	-1.19	—
K	3.9	—	Tb	-1.96	—
Ca	5.21	—	Dy	-1.16	—
Sc	2.02	—	Ho	-1.78	—
Ti	3.82	—	Er	-1.34	—
V	2.8	—	Tm	-2.16	—
Cr	4.51	—	Yb	-1.42	—
Mn	4.3	—	Lu	-2.16	—
Fe	6.37	—	Hf	-1.41	—
Co	3.86	—	Ta	-2.38	—
Ni	5.09	—	W	-1.41	—
Cu	3.06	—	Re	-2.0	—
Zn	2.3	—	Os	-0.86	—
Ga	0.78	—	Ir	-0.88	—
Ge	1.39	—	Pt	-0.64	—
As	0.04	—	Au	-1.34	—
Se	1.08	—	Hg	-1.09	—
Br	0.28	—	Tl	-1.36	—
Kr	0.99	—	Pb	-0.51	—
Rb	0.26	—	Bi	-1.61	—
Sr	0.61	—	Po	-99.0	—
Y	1.08	—	At	-99.0	—
Zr	1.45	—	Rn	-99.0	—
Nb	-0.8	—	Fr	-99.0	—
Mo	-0.38	—	Ra	-99.0	—
Tc	-99.0	—	Ac	-99.0	—
Ru	-0.51	—	Th	-2.2	—
Rh	-1.35	—	Pa	-99.0	—
Pd	-0.69	—	U	-2.8	—

NOTE—Relative Metal composition used where $a(\text{H}) = 12$. Composition measurements are taken from Milone et al. (2015a) populations A&E (P1 and P2 respectively). Where the relative composition is the the same for both P1 and P2; it is only listed in the P1 column for the sake of visual clarity.

References—Milone et al. (2015a)

We use the Dartmouth Stellar Evolution Program (DSEP, Dotter et al. 2008) to generate stellar models. DSEP is a one-dimensional stellar evolution code that includes a mixing length model of convection, gravitational settling, and diffusion. Using the solar composition presented in (Grevesse et al. 2007) (GAS07), MARCS model atmospheres, OPLIB high temperature opacities, and AESOPUS 2.0 low temperature opacities we find a solar calibrated mixing length parameter, α_{MLT} , of $\alpha_{MLT} = 1.901$. **Abundance measurements are derived from populations A&E in Milone et al. (2015a) (for P1 and P2 respectively).**

We use DSEP to evolve stellar models ranging in mass from 0.3 to 2.0 solar masses from the fully convective pre-main sequence to the tip of the red giant branch. Below $0.7 M_{\odot}$ we evolve a model every $0.03 M_{\odot}$ and above $0.7 M_{\odot}$ we evolve a model every $0.05 M_{\odot}$. We evolve models over a grid of mixing length parameters from $\alpha_{MLT} = 1.0$ to $\alpha_{MLT} = 2.0$ in steps of 0.1. For each mixing length, a grid of models and isochrones were calculated, with chemical compositions consistent with Milone et al. (2015a) populations A and E (see Tables 1 and 1) and a range of helium abundances ($Y=0.24, 0.27, 0.30, 0.33, 0.36, \text{ and } 0.39$). In total, 144 sets of isochrones, each with a unique composition and mixing length were calculated. Each model is evolved in DSEP with typical numeric tolerances of one part in 10^7 . Each model is allowed a maximum time step of 50 Myr.

For each combination of populations, Y , and α_{MLT} we use the isochrone generation code first presented in Dotter (2016) to generate a grid of isochrones. The isochrone generation code identified equivalent evolutionary points (EEPs) over a series of masses and interpolates between them. The grid of isochrones generated for this work is available as a digital supplement to this paper [10.5281/zenodo.10631439](https://doi.org/10.5281/zenodo.10631439). Given the complexity of the parameter space when fitting multiple populations, along with the recent warnings in the literature regarding overfitting datasets (e.g. Valle et al. 2022), we want to develop a more objective way of fitting isochrones to photometry than if we were to mark median ridge line positions by hand.

4. FIDANKA

When fitting isochrones to the clusters with multiple populations we have four main criteria for any method:

- The method must be robust enough to work along the entire main sequence, turn off, and much of the subgiant and red giant branch.

Table 2. Population Abundance Ratios

Population	[Fe/H]	[α /Fe]	[C/Fe]	[N/Fe]	[O/Fe]	[r/Fe]	[s/Fe]	C/O	X	Y	Z
A(1)	-1.13	0.32	-0.43	-0.28	0.31	-1.13	-1.13	0.10	0.7285	0.2700	0.00154
E(2)	-1.13	-0.11	-1.39	-0.02	-0.66	-1.13	-1.13	0.10	0.7594	0.240	0.00063

NOTE—Abundance ratios for populations P1 and P2 in NGC 2808.

References—Milone et al. (2015a)

- 305 • Any method should consider photometric uncer-
306 tainty in the fitting process.
- 307 • The method should be model independent, weight-
308 ing any n number of populations equally.
- 309 • The method should be automated and require
310 minimal intervention from the user.

311 We do not believe that any currently available soft-
312 ware is a match for our use case. Therefore, we have
313 developed our own software suite, *Fidanka*. *Fidanka*
314 is a Python package designed to automate much of the
315 process of measuring fiducial lines in CMDs, adhering to
316 the four criteria we lay out above. Primary features of
317 *Fidanka* may be separated into three categories: fidu-
318 cial line measurement, stellar population synthesis, and
319 isochrone optimization/fitting. Additionally, there are
320 utility functions that are detailed in the *Fidanka* docu-
321 mentation.

322 4.1. Fiducial Line Measurement

323 *Fidanka* takes an iterative approach to measuring fidu-
324 cial lines, the first step of which is to make a “guess”
325 as to the fiducial line. This initial guess is calculated
326 by splitting the CMD into magnitude bins, with uni-
327 form numbers of stars per bin (so that bins cover a
328 small magnitude range over densely populated regions
329 of the CMD, while covering a much larger magnitude
330 range in sparsely populated regions of the CMD, such
331 as the RGB). A unimodal Gaussian distribution is then
332 fit to the color distribution of each bin, and the resulting
333 mean color is used as the initial fiducial line guess. This
334 rough fiducial line will approximately trace the area of
335 highest density. The initial guess will be used to verti-
336 calize the CMD so that further algorithms can work in
337 1D magnitude bins without worrying about weighting
338 issues caused by varying projections of the evolution-
339 ary sequence onto the magnitude axis. Verticalization
340 is performed by taking the difference between the guess
341 fiducial line and the color of each star in the CMD.

342 If *Fidanka* were to simply apply the same algorithm
343 to the verticalized CMD, then the resulting fiducial line

344 would likely be a re-extraction of the initial fiducial
345 line guess. To avoid this outcome, we take a more
346 robust, number-density based approach that considers
347 the distribution of stars in both color and magnitude
348 space simultaneously. **As an example, in the case**
349 **of this work**, for each star in the CMD we first use
350 an `introslect` partitioning algorithm to select the 50
351 nearest stars in F814W vs. F275W-F814W space. **It**
352 **should be noted that unlike methods using chro-**
353 **mosome maps *Fidanka* only considers two filters**
354 **and therefore might lose access to information**
355 **better traced by other filters.** To account for the
356 case where the star is at an extreme edge of the CMD,
357 those 50 stars include the star itself (such that we really
358 select 49 stars + 1). We use `qhull`¹ (Barber et al. 1996)
359 to calculate the convex hull of those 50 points. The
360 number density at each star then is defined as $50/A_{hull}$,
361 where A_{hull} is the area of the convex hull. Because we
362 use a fixed number of points per star, and a partition-
363 ing algorithm as opposed to a sorting algorithm, this
364 method scales like $\mathcal{O}(n)$, where n is the number of stars
365 in the CMD. This method also intrinsically weights the
366 density of each star equally, as the counting statistics
367 per bin are uniform. We are left with a CMD in which
368 each star has a defined number density (Figure 1).

369 *Fidanka* can now exploit this density map to fit a
370 better fiducial line to the data, as the density map is far
371 more robust to outliers. There are multiple algorithms
372 that we implement to fit the fiducial line to the color-
373 density profile in each magnitude bin (Figure 2); these
374 are explained in more detail in the *Fidanka* documen-
375 tation. However, of most relevance here is the Bayesian
376 Gaussian Mixture Modeling (BGMM) method. BGMM
377 is a clustering algorithm that, for some fixed number
378 of n-dimensional Gaussian distributions, K , determines
379 the mean, covariance, and mixing probability (some-
380 what analogous to amplitude) of each k^{th} distribution,
381 such that the local lower bound of the likelihood of each

¹ <https://www.qhull.com>

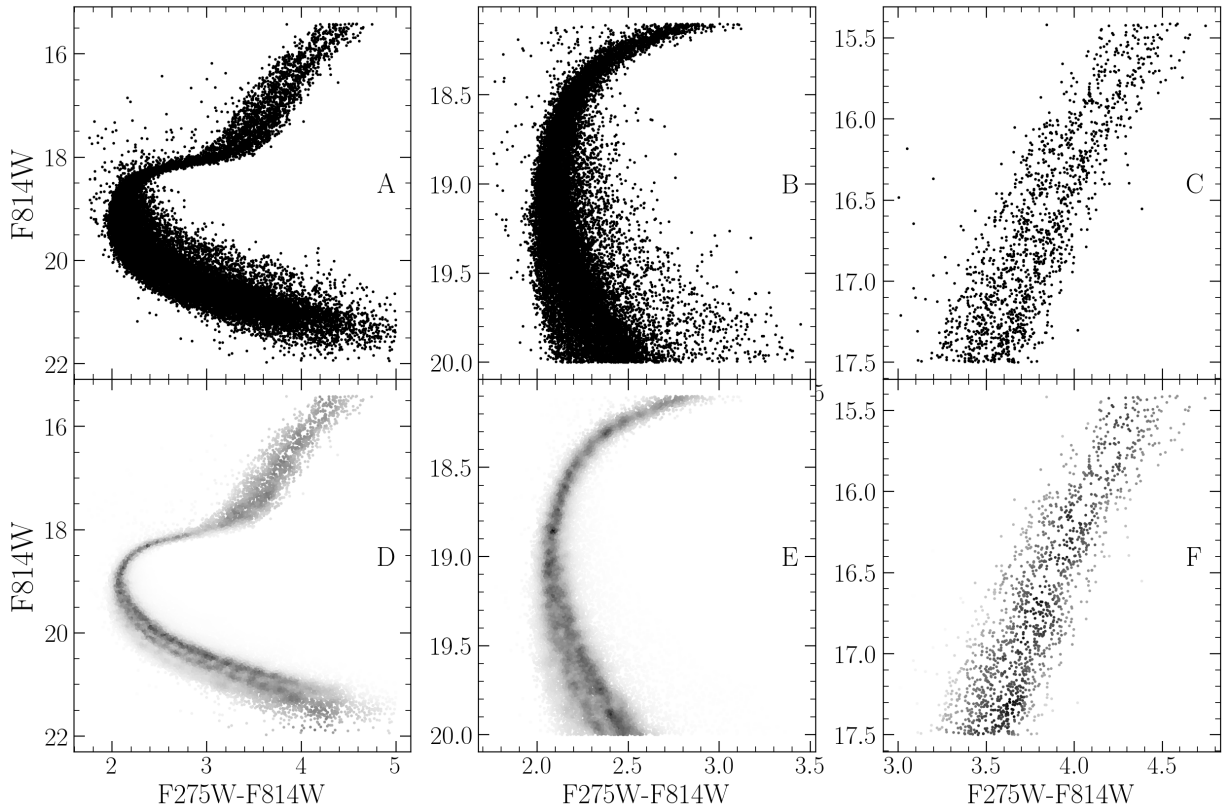


Figure 1. Figures in the top row are the raw CMD, while figures in the bottom row are colored by the density map. Density map demo showing density estimate over different parts of the evolutionary sequence. The left panel shows the density map over the entire evolutionary sequence, while the middle panel shows the density map over the main sequence and the right panel shows the density map over the RGB.

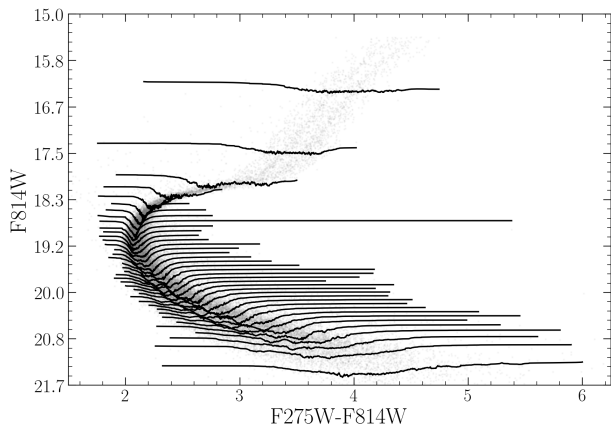


Figure 2. CMD where point brightness is determined by local density. Lines show the density-color profile in each magnitude bin. In this figure adaptive binning targeted 1000 stars per bin

382 star belonging strongly to a single distribution is maxi-
383 mized.

384 Maximization is performed using the Dirichlet pro-
385 cess, which is a non-parametric Bayesian method of de-
386 termining the number of Gaussian distributions, K , that

387 best fit the data (Ferguson 1973; Pedregosa et al. 2011).
388 Use of the Dirichlet process allows for dynamic varia-
389 tion in the number of inferred populations from mag-
390 nitude bin to magnitude bin. Specifically, populations
391 are clearly visually separated from the lower main se-
392 quence through the turn off; however, at the turn off
393 and throughout much of the subgiant branch, the two
394 visible populations overlap due to their similar ages (i.e.
395 Jordán et al. 2002). The Dirichlet process allows for the
396 BGMM method to infer a single population in these re-
397 gions, while inferring two populations in regions where
398 they are clearly separated. More generally, the use of
399 the Dirichlet process removes the need for a prior on
400 the exact number of populations to fit. Rather, the user
401 specifies a upper bound on the number of populations
402 within the cluster. An example bin ($F814W = 20.6$) is
403 shown in Figure 3.

404 Fidanka's BGMM method first breaks down the verti-
405 calized CMD into magnitude bins with uniform num-
406 bers of stars per bin (here we adopt 250). Any stars
407 left over are placed into the final bin. For each bin a
408 BGMM model with a maximum of five populations is
409 fit to the color density profile. The number of popula-

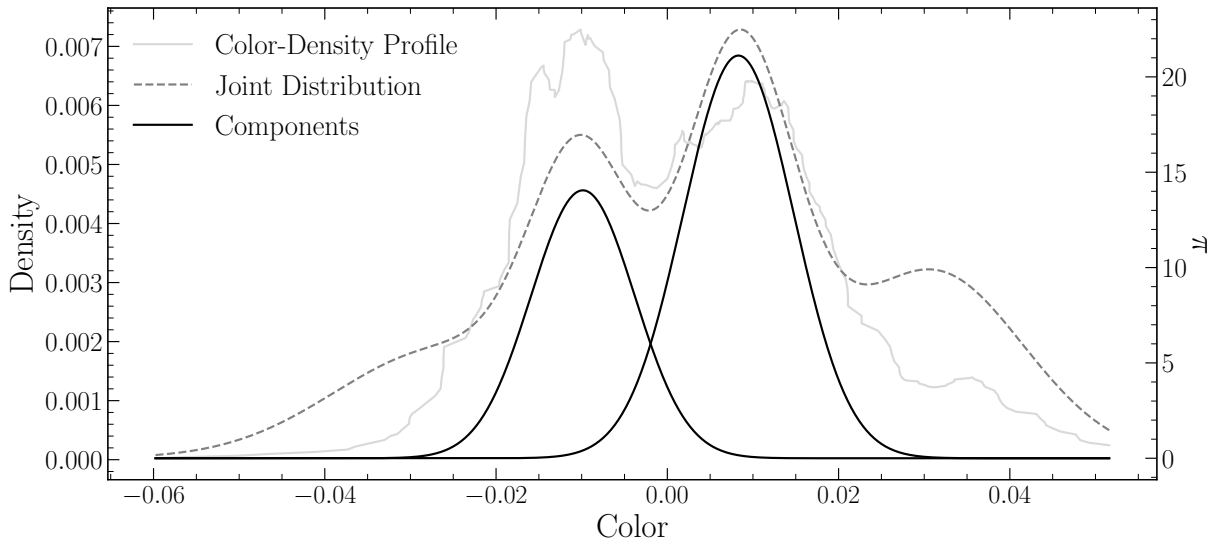


Figure 3. Example of BGMM fit to a magnitude bin. The grey line shows the underlying color-density profile, while the black dashed line shows the joint distribution of each BGMM component. The solid black lines show the two selected components.

410 tions is then inferred from the weighting parameter (the
 411 mixing probability) of each population. If the weight-
 412 ing parameter of any k^{th} components is less than **0.05**,
 413 then that component is considered to be spurious and
 414 removed. Additionally, if the number of populations in
 415 the bin above and the bin below are the same, then the
 416 number of populations in the current bin is forced to be
 417 the same as the number of populations in the bin above.
 418 Finally, the initial guess fiducial line is added back to
 419 the BGMM inferred line. Figure 4 shows the resulting
 420 fiducial line(s) in each magnitude bin for both a verti-
 421 calized CMD and a non-verticalized CMD. In contrast
 422 to other work in the literature where evidence for up to
 423 five distinct populations has been found, we only find
 424 evidence for two stellar populations.

425 This method of fiducial line extraction effectively dis-
 426 criminates between multiple populations along the main
 427 sequence and RGB of a cluster, while simultaneously al-
 428 lowing for the presence of a single population along the
 429 MSTO and subgiant branch.

430 We can adapt this density map-based BGMM method
 431 to consider photometric uncertainties by adopting a sim-
 432 ple Monte Carlo approach. Instead of measuring the
 433 fiducial line(s) a single time, **Fidanka** can measure the
 434 fiducial line(s) many times, resampling the data with
 435 replacements each time. For each resampling, **Fidanka**
 436 adds a random offset to each filter based on the photo-
 437 metric uncertainties of each star. From these n measure-
 438 ments the mean fiducial line for each sequence can be
 439 identified along with upper and lower-bound confidence
 440 intervals in each magnitude bin.

4.2. Stellar Population Synthesis

442 While not extensively used in this paper **Fidanka** can,
 443 in addition to measuring fiducial lines, perform stellar
 444 population synthesis. **Fidanka**'s population synthesis
 445 module can generate synthetic stellar populations from
 446 a set of MIST-formatted isochrones. This is of primary
 447 importance for binary population modeling. The mod-
 448 ule is also used to generate synthetic CMDs for the pur-
 449 pose of testing the fiducial line extraction algorithms
 450 against priors.

451 **Fidanka** uses MIST-formatted isochrones (Dotter
 452 2016) as input along with distance modulus, B-V color
 453 excess, binary mass fraction, and bolometric corrections.
 454 An arbitrarily large number of isochrones may be used
 455 to define an arbitrary number of populations. Synthetic
 456 stars are samples from each isochrone based on a defin-
 457 able probability; For example, it is believed that $\sim 90\%$
 458 of stars in globular clusters are younger population (e.g.
 459 Suntzeff & Kraft 1996; Carretta 2013). Based on the
 460 metallicity, μ , and E(B-V) of each isochrone, bolometric
 461 corrections are taken from bolometric correction tables.
 462 Where bolometric correction tables do not include ex-
 463 act metallicities or extinctions a linear interpolation is
 464 performed between the two bounding values.

4.3. Isochrone Optimization

465
 466 The optimization routines in **Fidanka** will find the
 467 best fit distance modulus, B-V color excess, and binary
 468 number fraction for a given set of isochrones. If a sin-
 469 gle isochrone is provided then the optimization is done
 470 by minimizing the χ^2 of the perpendicular distances
 471 between an isochrone and a fiducial line. If multiple
 472 isochrones are provided then those isochrones are first
 473 used to run a stellar population synthesis and gener-

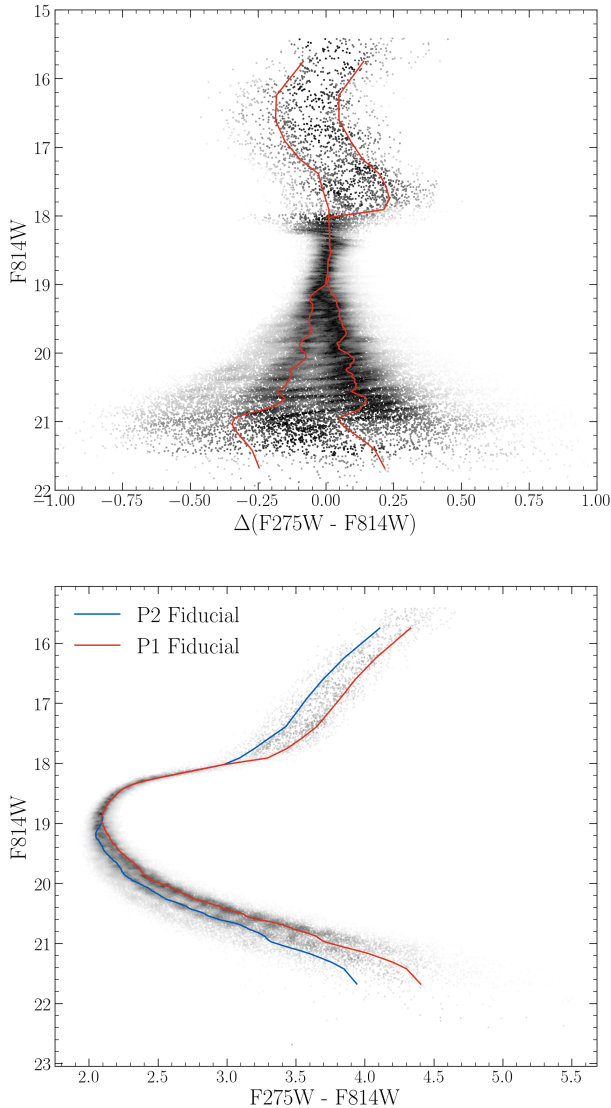


Figure 4. Verticalized CMD (where the color of each data point is subtracted from the color of the fiducial line at that magnitude) where point brightness is determined by density (top). CMD where point brightness is determined by density, calculated fiducial lines are shown (bottom). The data used is from the Hubble Space Telescope UV Legacy Survey of Galactic Globular Clusters.

474 ate a synthetic CMD. The optimization is then done by
 475 minimizing the χ^2 of both the perpendicular distances
 476 between and widths of the observed fiducial line and the
 477 fiducial line of the synthetic CMD.

478 4.4. *Fidanka* Testing

479 In order to validate *Fidanka* we have run a series of
 480 injection recovery tests using *Fidanka*'s population syn-
 481 thesis routines to build various synthetic populations
 482 and *Fidanka*'s fiducial measurement routines to recover
 483 these populations. Each population was generated us-

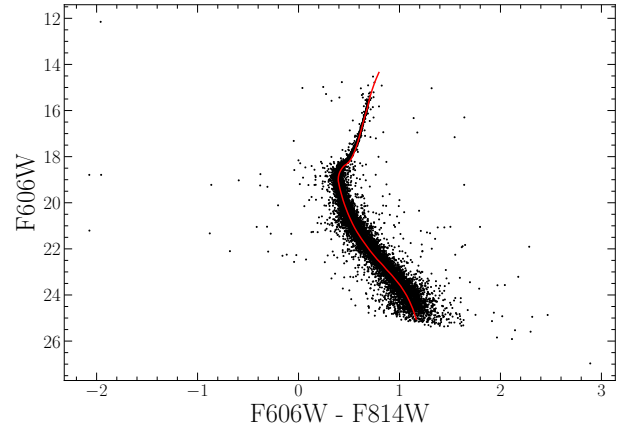


Figure 5. Synthetic population generated by *Fidanka* at 10000pc with $E(B-V) = 0$, and an age of 12 Gyr along with the best fitting isochrone. The best fit parameters are derived to be $\mu = 15.13$, $E(B-V)=0.001$, and an age of 12.33 Gyr.

484 ing the initial mass function given in (Milone et al. 2012)
 485 for the redmost population ($\alpha = -1.2$). Further, every
 486 population was given a binary population fraction of
 487 10%, distance uniformly sampled between 5000pc and
 488 15000pc, and a B-V color excess uniformly sampled be-
 489 tween 0 and 0.1. *Fidanka* makes use of ACS arti-
 490 ficial star tests (Anderson et al. 2008) to model
 491 the noise and completeness of a synthetic popu-
 492 lation in passbands covered by those tests. Full
 493 details on how *Fidanka* uses artificial star tests
 494 may be found on its documentation page² Finally,
 495 each synthetic population was generated using a fixed
 496 age uniformly sampled between 7 Gyr and 14 Gyr. An
 497 example synthetic population, along with its associated
 498 best fit isochrone, are shown in Figure 5.

499 For each trial we use *Fidanka* to measure the fiducial
 500 line and then optimize that fiducial line against the orig-
 501 inating isochrone to estimate distance modulus, age, and
 502 color B-V excess. Figure 6 is built from 1000 Monte-
 503 Carlo trials and shows the mean and width of the per-
 504 cent error distributions for μ , A_v , and age. In general
 505 *Fidanka* is able to recover distance moduli effectively
 506 with age and $E(B-V)$ recovery falling in line with other
 507 literature that does not consider the CMD outside of the
 508 main sequence, main sequence turn off, subgiant, and
 509 red giant branches. Specifically, it should be noted that
 510 *Fidanka* is not set up to model the horizontal branch.

511 5. ISOCHRONE FITTING

512 We fit pairs of isochrones to the HUGS data for NGC
 513 2808 using *Fidanka*, as described in §4. As was men-

² <https://tboudreaux.github.io/fidanka/>

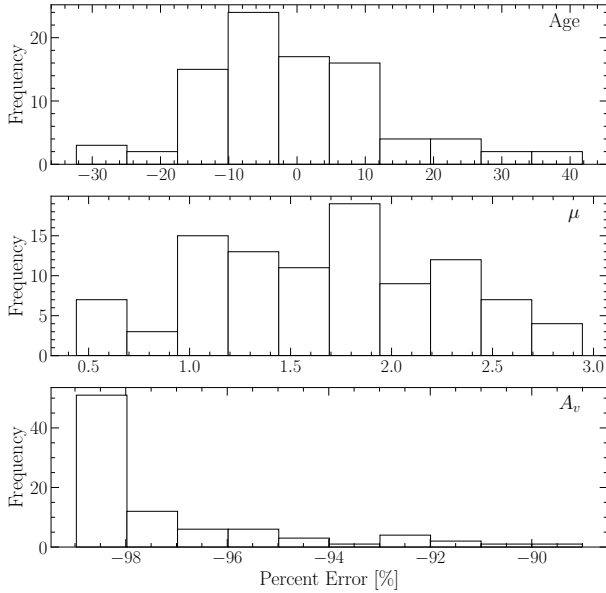


Figure 6. Percent error distribution for each of the three deriver parameters. Note that these values will be sensitive to the magnitude uncertainties of the photometry. Here we made use of the ACS artificial star tests to estimate the uncertainties.

tioned in §4.1, the method used by Fidanka only considers two filters — in the case of this work F275W and F814W — and therefore might be unable to distinguish between populations separated only in the higher-dimensional space of a chromosome map. For further discussion of why we adopt this method, despite it limits, we refer the reader to §5.1. Two isochrones, one for P1 and one for P2 are fit simultaneously. These isochrones are constrained to have distance modulus, μ , and color excess, $E(B-V)$ which agree to within 0.5% and an ages which agree to within 1%. Moreover, we constrain the mixing length, α_{ML} , for any two isochrones in a set to be within 0.5 of one and other. For each isochrone set we optimize μ_{P1} , μ_{P2} , $E(B-V)_{P1}$, $E(B-V)_{P2}$, Age_{P1} , and Age_{P2} in order to reduce the χ^2 distance ($\chi^2 = \sum \sqrt{\Delta color^2 + \Delta mag^2}$) between the fiducial lines and the isochrones. Because we fit fiducial lines directly, we do not need to consider the binary population fraction, f_{bin} , as a free parameter.

The best fit isochrones are shown in Figure 7 and optimized parameters for these are presented in Table 1. The initial guess for the age of these populations was locked to 12 Gyr and the initial extinction was locked to 0.5 mag. The initial guess for the distance modulus was determined at run time using a dynamic time warping algorithm to best align the morphologies of the fiducial line with

the target isochrone. This algorithm is explained in more detail in the Fidanka documentation under the function called `guess_mu`. We find helium mass fractions that are consistent with those identified in past literature (e.g. Milone et al. 2015a). Note that our helium mass fraction grid has a spacing of 0.03 between grid points and we are therefore unable to resolve between certain proposed helium mass fractions for the younger sequence (for example between 0.37 and 0.39). We also note that the best fit mixing length parameters which we derive for P1 and P2 do not agree within their uncertainties. This is not surprising, as the much higher mean molecular mass of P2 — when compared to P1, due to population P2’s larger helium mass fraction — will result in a steeper adiabatic temperature gradient.

Past literature (e.g. Milone et al. 2015a, 2018) has found helium mass fraction variation from the redmost to bluemost populations of ~ 0.12 . Here we find a helium mass fraction variation of 0.15 that, given the spacing of the helium grid we use is consistent with these past results. The helium mass fractions we derive for P1 and P2 are consistent with those of populations A and E in Milone et al. (2015a); however, populations B+C and D in that study are more prominently separated in the F275W-F814W colorband. The inferred helium mass fractions for P1 and P2 are not consistent with those reported for populations B+C and D.

5.1. The Number of Populations in NGC 2808

In order to estimate the number of populations that ideally fit the NGC 2808 F275W-F814W photometry without overfitting the data we make use of silhouette analysis (Rousseeuw 1987; Shahapure & Nicholas 2020, and in a similar manner to how Valle et al. (2022) perform their analysis of spectroscopic data). We find the average silhouette score for the hypothesizes that there are two, three, four, or five population in each magnitude bin. We preform this analysis over all magnitude using routines built into the standard Python module `sklearn`. Figure 8 (top) shows the silhouette analysis results and that two populations fit the photometry most ideally. This result is in line with what our BGMM model predicts for the majority of the CMD.

While we make use of a purely CMD-based approach in this work, other literature has made use of chromosome maps. These consist of implicitly verticalized pseudo colors. In the chromosome map for NGC 2808 there may be evidence for more than two populations; further, the chromosome maps used include in-

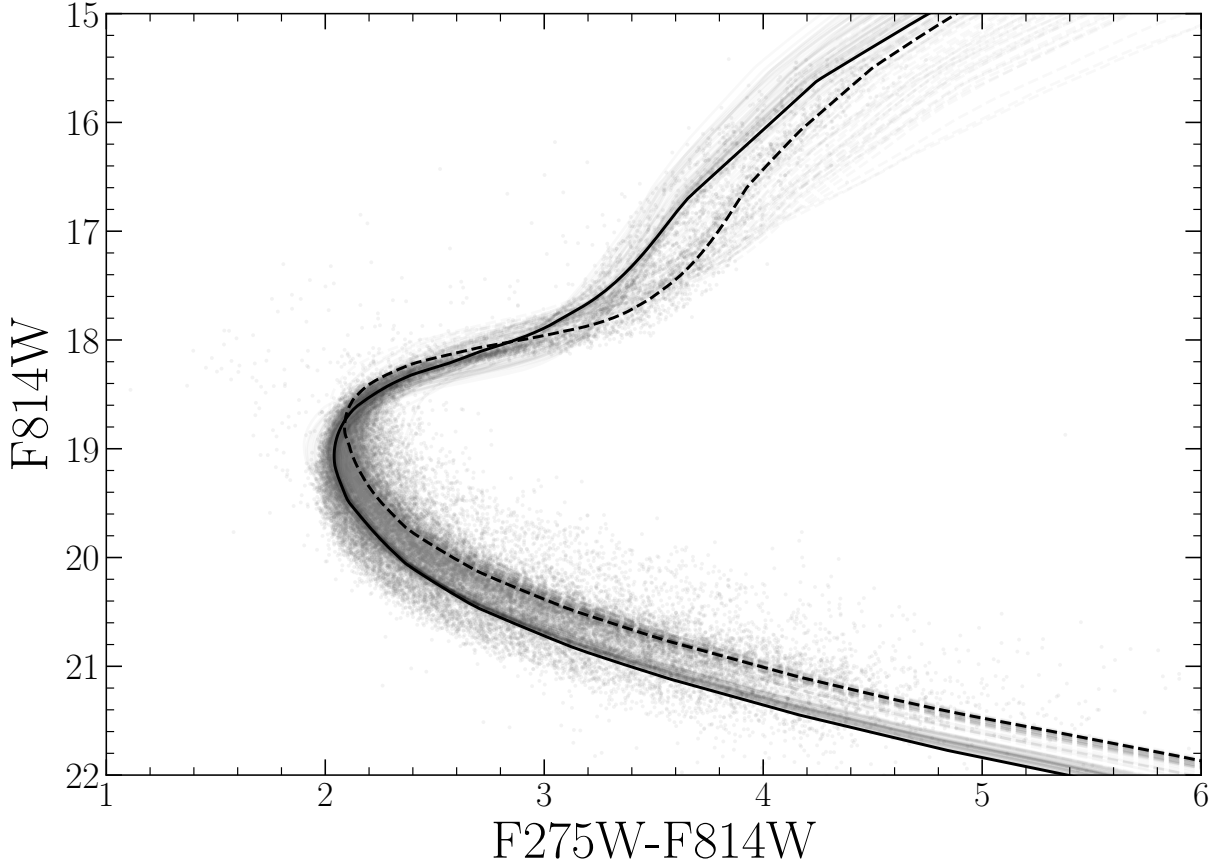


Figure 7. Best fit isochrone results for NGC 2808. The best fit P1 and P2 models are shown as black lines. The following 50 best fit models are presented as grey lines. The dashed black line is fit to P1, while the solid black line is fit to P2.

Population	Age [Gyr]	Distance Modulus	Extinction [mag]	Y	α_{ML}	χ^2_ν
P1	$12.996^{+0.87}_{-0.64}$	15.021	0.54	0.24	2.050	0.021
P2	$13.061^{+0.86}_{-0.69}$	15.007	0.537	0.39	1.600	0.033

Table 1. Best fit parameters derived from fitting isochrones to the fiducial lines derived from the NCG 2808 photometry. The one sigma uncertainty reported on population age were determined from the 16th and 84th percentiles of the distribution of best fit isochrones ages.

593 formation from additional filters (F336W and
 594 F438W) which we do not use in our CMD ap-
 595 proach. We perform the same analysis on the
 596 F336W-F438W CMD using Fidanka as we do
 597 on the F275W-F814W CMD. While the cluster-
 598 ing algorithm does find a more strongly distin-
 599 guished potential third population using these
 600 filters (Figure 8 bottom), the two population
 601 hypothesis remains strongly preferred. More-
 602 over, the process of transforming magnitude
 603 measurements into chromosome space results
 604 in dramatically increased uncertainties for each
 605 star. We find a mean fractional uncertainty

606 for chromosome parameters — $\Delta_{F275W,F814W}$ and
 607 $\Delta_{CF275W,F336W,F438W}$ — of ≈ 1 (Figure 9) when
 608 starting with magnitude measurements having a
 609 mean best-case (i.e. where the uncertainty is
 610 assumed to only be due to Poisson statistics)
 611 fractional uncertainty of ≈ 0.0005 . Fractional un-
 612 certainties for chromosome parameters were cal-
 613 culated via standard propagation of uncertainty.
 614 Because of how Fidanka operates, i.e. resampling
 615 a probability distribution for each star in order
 616 to identify clusters, we are unable to make sta-
 617 tistically meaningful statements from the chro-
 618 mosome map

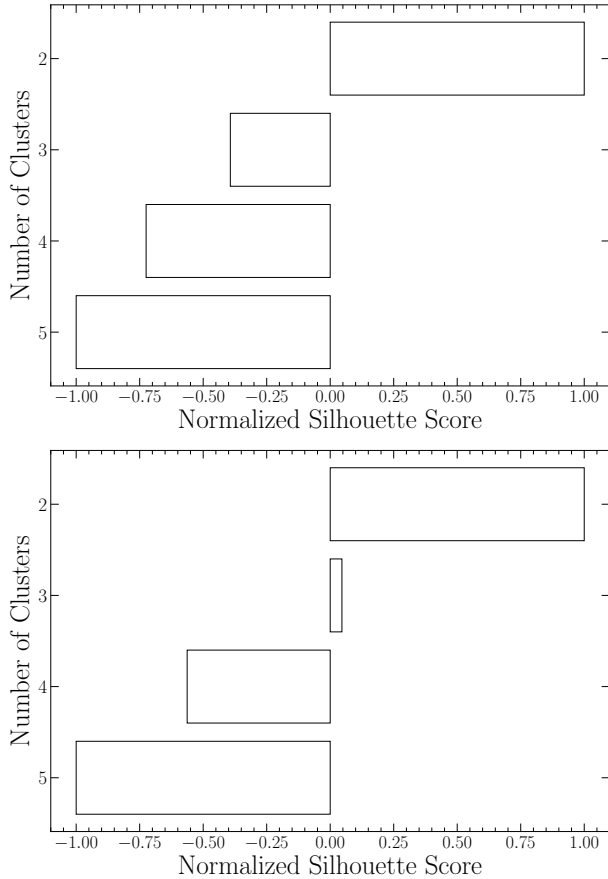


Figure 8. Silhouette analysis for NGC 2808 F275W-F814W (top) and F336W-F438W (bottom) photometry. The Silhouette scores are an average of score for each magnitude bin. Scores have been normalized to indicate the most well-distinguished (+1) to least well-distinguished (-1) hypotheses.

6. CONCLUSION

Here we have performed the first chemically self-consistent modeling of the Milky Way Globular Cluster NGC 2808. We find that, updated atmospheric boundary conditions and opacity tables do not have a significant effect on the inferred helium abundances of multiple populations. Specifically, we find that P1 has a helium mass fraction of 0.24, while P2 has a helium mass fraction of 0.39. Additionally, we find that the ages of these two populations agree within uncertainties. We only find evidence for two distinct stellar populations, which is in agreement with recent work studying the number of populations in NGC 2808 spectroscopic data.

We introduce a new software suite for globular cluster science, **Fidanka**, which has been released under a permissive open source license. **Fidanka** aims to provide a statistically robust set of tools for estimating the parameters of multiple populations within globular clusters.

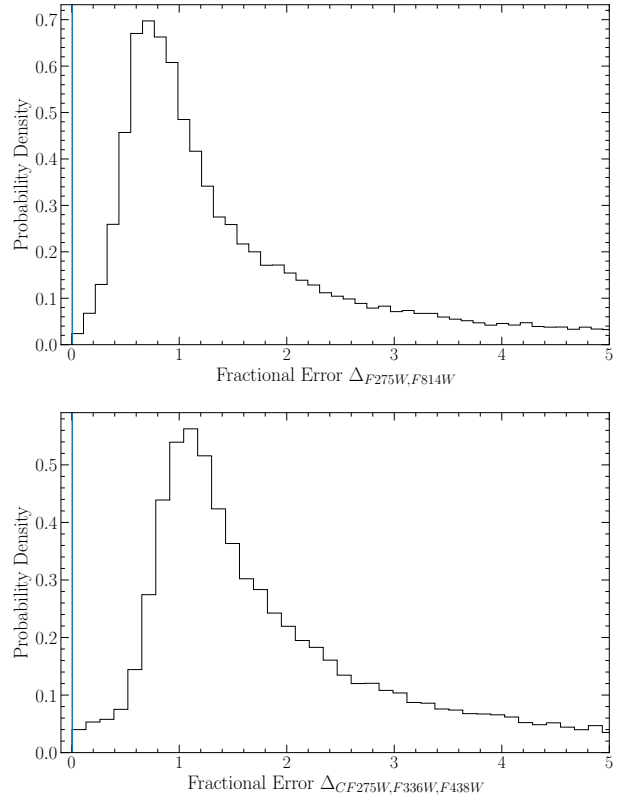


Figure 9. Fractional error distribution of $\Delta_{F275W, F814W}$ (top) and $\Delta_{CF275W, F336W, F438W}$. The vertical line near 0 in each figure indicates the mean fractional error of the magnitude measurements used to find the chromosome parameters.

This work has made use of the NASA astrophysical data system (ADS). We would like to thank Elisabeth Newton and Aaron Dotter for their support and for useful discussion related to the topic of this paper. Additionally, we would like to thank Kara Fagerstrom, Aylin Garcia Soto, and Keighley Rockcliffe for their useful discussion related to in this work. We acknowledge the support of a NASA grant (No. 80NSSC18K0634).

REFERENCES

Alcaino, G. 1975, *A&AS*, 21, 15

Anderson, J., Sarajedini, A., Bedin, L. R., et al. 2008, *AJ*, 135, 2055, doi: [10.1088/0004-6256/135/6/2055](https://doi.org/10.1088/0004-6256/135/6/2055)

- 648 Barber, C. B., Dobkin, D. P., & Huhdanpaa, H. 1996, ACM
649 Transactions on Mathematical Software (TOMS), 22, 469
650 Bastian, N., & Lardo, C. 2015, MNRAS, 453, 357,
651 doi: [10.1093/mnras/stv1661](https://doi.org/10.1093/mnras/stv1661)
652 Bastian, N., & Lardo, C. 2018, Annual Review of
653 Astronomy and Astrophysics, 56, 83
654 Bekki, K., & Chiba, M. 2002, The Astrophysical Journal,
655 566, 245, doi: [10.1086/337984](https://doi.org/10.1086/337984)
656 Boudreaux, E. M., & Chaboyer, B. C. 2023, ApJ, 944, 129,
657 doi: [10.3847/1538-4357/acb685](https://doi.org/10.3847/1538-4357/acb685)
658 Boylan-Kolchin, M. 2018, MNRAS, 479, 1423,
659 doi: [10.1093/mnras/sty1490](https://doi.org/10.1093/mnras/sty1490)
660 Brodie, J. P., & Strader, J. 2006, Annu. Rev. Astron.
661 Astrophys., 44, 193
662 Brown, J. H., Burkert, A., & Truran, J. W. 1991, ApJ, 376,
663 115, doi: [10.1086/170260](https://doi.org/10.1086/170260)
664 —. 1995, ApJ, 440, 666, doi: [10.1086/175304](https://doi.org/10.1086/175304)
665 Cadelano, M., Dalessandro, E., Salaris, M., et al. 2022,
666 ApJL, 924, L2, doi: [10.3847/2041-8213/ac424a](https://doi.org/10.3847/2041-8213/ac424a)
667 Carretta, E. 2006, AJ, 131, 1766, doi: [10.1086/499565](https://doi.org/10.1086/499565)
668 —. 2013, A&A, 557, A128,
669 doi: [10.1051/0004-6361/201322103](https://doi.org/10.1051/0004-6361/201322103)
670 —. 2015, ApJ, 810, 148, doi: [10.1088/0004-637X/810/2/148](https://doi.org/10.1088/0004-637X/810/2/148)
671 Carretta, E., Bragaglia, A., & Cacciari, C. 2004, ApJL,
672 610, L25, doi: [10.1086/423034](https://doi.org/10.1086/423034)
673 Carretta, E., Bragaglia, A., Gratton, R. G., et al. 2010,
674 Astronomy & Astrophysics, 516, A55
675 Colgan, J., Kilcrease, D. P., Magee, N. H., et al. 2016, in
676 APS Meeting Abstracts, Vol. 2016, APS Division of
677 Atomic, Molecular and Optical Physics Meeting
678 Abstracts, D1.008
679 de Mink, S. E., Pols, O. R., Langer, N., & Izzard, R. G.
680 2009, A&A, 507, L1, doi: [10.1051/0004-6361/200913205](https://doi.org/10.1051/0004-6361/200913205)
681 Decressin, T., Meynet, G., Charbonnel, C., Prantzos, N., &
682 Ekström, S. 2007, A&A, 464, 1029,
683 doi: [10.1051/0004-6361:20066013](https://doi.org/10.1051/0004-6361:20066013)
684 Denissenkov, P. A., & Hartwick, F. D. A. 2014, MNRAS,
685 437, L21, doi: [10.1093/mnras/slt133](https://doi.org/10.1093/mnras/slt133)
686 D’Ercole, A., D’Antona, F., Ventura, P., Vesperini, E., &
687 McMillan, S. L. W. 2010, MNRAS, 407, 854,
688 doi: [10.1111/j.1365-2966.2010.16996.x](https://doi.org/10.1111/j.1365-2966.2010.16996.x)
689 D’Ercole, A., Vesperini, E., D’Antona, F., McMillan, S.
690 L. W., & Recchi, S. 2008, MNRAS, 391, 825,
691 doi: [10.1111/j.1365-2966.2008.13915.x](https://doi.org/10.1111/j.1365-2966.2008.13915.x)
692 Dotter, A. 2016, ApJS, 222, 8,
693 doi: [10.3847/0067-0049/222/1/8](https://doi.org/10.3847/0067-0049/222/1/8)
694 Dotter, A., Chaboyer, B., Jevremović, D., et al. 2008, The
695 Astrophysical Journal Supplement Series, 178, 89
696 Dotter, A., Ferguson, J. W., Conroy, C., et al. 2015,
697 MNRAS, 446, 1641, doi: [10.1093/mnras/stu2170](https://doi.org/10.1093/mnras/stu2170)
698 Ferguson, T. S. 1973, The annals of statistics, 209
699 Gratton, R., Sneden, C., & Carretta, E. 2004, ARA&A, 42,
700 385, doi: [10.1146/annurev.astro.42.053102.133945](https://doi.org/10.1146/annurev.astro.42.053102.133945)
701 Gratton, R. G., Carretta, E., & Bragaglia, A. 2012,
702 Astronomy and Astrophysics Reviews, 20, 50,
703 doi: [10.1007/s00159-012-0050-3](https://doi.org/10.1007/s00159-012-0050-3)
704 Gratton, R. G., Lucatello, S., Carretta, E., et al. 2011,
705 A&A, 534, A123, doi: [10.1051/0004-6361/201117690](https://doi.org/10.1051/0004-6361/201117690)
706 Grevesse, N., Asplund, M., & Sauval, A. J. 2007, SSRv,
707 130, 105, doi: [10.1007/s11214-007-9173-7](https://doi.org/10.1007/s11214-007-9173-7)
708 Gustafsson, B., Edvardsson, B., Eriksson, K., et al. 2008,
709 A&A, 486, 951, doi: [10.1051/0004-6361:200809724](https://doi.org/10.1051/0004-6361:200809724)
710 Hong, S., Lim, D., Chung, C., et al. 2021, AJ, 162, 130,
711 doi: [10.3847/1538-3881/ac0ce6](https://doi.org/10.3847/1538-3881/ac0ce6)
712 Hudson, M. J., & Robison, B. 2018, Monthly Notices of the
713 Royal Astronomical Society, 477, 3869,
714 doi: [10.1093/mnras/sty844](https://doi.org/10.1093/mnras/sty844)
715 Husser, T. O., Wende-von Berg, S., Dreizler, S., et al. 2013,
716 A&A, 553, A6, doi: [10.1051/0004-6361/201219058](https://doi.org/10.1051/0004-6361/201219058)
717 Jordán, A., Côté, P., West, M. J., & Marzke, R. O. 2002,
718 ApJL, 576, L113, doi: [10.1086/343759](https://doi.org/10.1086/343759)
719 Kalirai, J. S., & Richer, H. B. 2010, Philosophical
720 Transactions of the Royal Society of London Series A,
721 368, 755, doi: [10.1098/rsta.2009.0257](https://doi.org/10.1098/rsta.2009.0257)
722 Kostogryz, N., Shapiro, A. I., Witzke, V., et al. 2023,
723 Research Notes of the AAS, 7, 39,
724 doi: [10.3847/2515-5172/acc180](https://doi.org/10.3847/2515-5172/acc180)
725 Kravtsov, A. V., & Gnedin, O. Y. 2005, The Astrophysical
726 Journal, 623, 650
727 Kurucz, R.-L. 1993, Kurucz CD-Rom, 13
728 Lardo, C., Salaris, M., Cassisi, S., & Bastian, N. 2022,
729 A&A, 662, A117, doi: [10.1051/0004-6361/202243843](https://doi.org/10.1051/0004-6361/202243843)
730 Latour, M., Husser, T. O., Giesers, B., et al. 2019, A&A,
731 631, A14, doi: [10.1051/0004-6361/201936242](https://doi.org/10.1051/0004-6361/201936242)
732 Legnardi, M. V., Milone, A. P., Armillotta, L., et al. 2022,
733 MNRAS, 513, 735, doi: [10.1093/mnras/stac734](https://doi.org/10.1093/mnras/stac734)
734 Marigo, P., & Aringer, B. 2009, A&A, 508, 1539,
735 doi: [10.1051/0004-6361/200912598](https://doi.org/10.1051/0004-6361/200912598)
736 Marigo, P., Aringer, B., Girardi, L., & Bressan, A. 2022,
737 ApJ, 940, 129, doi: [10.3847/1538-4357/ac9b40](https://doi.org/10.3847/1538-4357/ac9b40)
738 Marino, A. F., Milone, A. P., Karakas, A. I., et al. 2015,
739 Monthly Notices of the Royal Astronomical Society, 450,
740 815, doi: [10.1093/mnras/stv420](https://doi.org/10.1093/mnras/stv420)
741 Martocchia, S., Dalessandro, E., Lardo, C., et al. 2019,
742 Monthly Notices of the Royal Astronomical Society, 487,
743 5324, doi: [10.1093/mnras/stz1596](https://doi.org/10.1093/mnras/stz1596)
744 Milone, A. P., Piotto, G., Bedin, L. R., et al. 2012, ApJ,
745 744, 58, doi: [10.1088/0004-637X/744/1/58](https://doi.org/10.1088/0004-637X/744/1/58)
746 Milone, A. P., Marino, A. F., Piotto, G., et al. 2015a, ApJ,
747 808, 51, doi: [10.1088/0004-637X/808/1/51](https://doi.org/10.1088/0004-637X/808/1/51)

- 748 —. 2015b, MNRAS, 447, 927, doi: [10.1093/mnras/stu2446](https://doi.org/10.1093/mnras/stu2446)
- 749 Milone, A. P., Piotto, G., Renzini, A., et al. 2017, MNRAS,
750 464, 3636, doi: [10.1093/mnras/stw2531](https://doi.org/10.1093/mnras/stw2531)
- 751 Milone, A. P., Marino, A. F., Renzini, A., et al. 2018,
752 MNRAS, 481, 5098, doi: [10.1093/mnras/sty2573](https://doi.org/10.1093/mnras/sty2573)
- 753 Pasquato, M., & Milone, A. 2019, arXiv e-prints,
754 arXiv:1906.04983, doi: [10.48550/arXiv.1906.04983](https://doi.org/10.48550/arXiv.1906.04983)
- 755 Pedregosa, F., Varoquaux, G., Gramfort, A., et al. 2011,
756 Journal of Machine Learning Research, 12, 2825
- 757 Peebles, P. J. E., & Dicke, R. H. 1968, ApJ, 154, 891,
758 doi: [10.1086/149811](https://doi.org/10.1086/149811)
- 759 Peng, E. W., Ferguson, H. C., Goudfrooij, P., et al. 2011,
760 The Astrophysical Journal, 730, 23
- 761 Piotto, G. 2018, HST UV Globular Cluster Survey
762 ("HUGS"), STScI/MAST, doi: [10.17909/T9810F](https://doi.org/10.17909/T9810F)
- 763 Piotto, G., Bedin, L. R., Anderson, J., et al. 2007, The
764 Astrophysical Journal Letters, 661, L53,
765 doi: [10.1086/518503](https://doi.org/10.1086/518503)
- 766 Piotto, G., Milone, A. P., Bedin, L. R., et al. 2015, AJ, 149,
767 91, doi: [10.1088/0004-6256/149/3/91](https://doi.org/10.1088/0004-6256/149/3/91)
- 768 Plez, B. 2008, Physica Scripta Volume T, 133, 014003,
769 doi: [10.1088/0031-8949/2008/T133/014003](https://doi.org/10.1088/0031-8949/2008/T133/014003)
- 770 Prantzos, N., Charbonnel, C., & Iliadis, C. 2007, A&A, 470,
771 179, doi: [10.1051/0004-6361:20077205](https://doi.org/10.1051/0004-6361:20077205)
- 772 Renzini, A. 2008, Monthly Notices of the Royal
773 Astronomical Society, 391, 354,
774 doi: [10.1111/j.1365-2966.2008.13892.x](https://doi.org/10.1111/j.1365-2966.2008.13892.x)
- 775 Richer, H. B., Fahlman, G. G., Buonanno, R., et al. 1991,
776 ApJ, 381, 147, doi: [10.1086/170637](https://doi.org/10.1086/170637)
- 777 Rousseeuw, P. J. 1987, Journal of Computational and
778 Applied Mathematics, 20, 53,
779 doi: [https://doi.org/10.1016/0377-0427\(87\)90125-7](https://doi.org/10.1016/0377-0427(87)90125-7)
- 780 Salaris, M., & Cassisi, S. 2005, Evolution of stars and
781 stellar populations (John Wiley & Sons)
- 782 Sandage, A. R. 1953, AJ, 58, 61, doi: [10.1086/106822](https://doi.org/10.1086/106822)
- 783 Shahapure, K. R., & Nicholas, C. 2020, in 2020 IEEE 7th
784 International Conference on Data Science and Advanced
785 Analytics (DSAA), 747–748,
786 doi: [10.1109/DSAA49011.2020.00096](https://doi.org/10.1109/DSAA49011.2020.00096)
- 787 Smith, G. H. 1987, Publications of the Astronomical
788 Society of the Pacific, 99, 67, doi: [10.1086/131958](https://doi.org/10.1086/131958)
- 789 Sneden, C., Kraft, R. P., Prosser, C. F., & Langer, G. 1992,
790 The Astronomical Journal, 104, 2121
- 791 Suntzeff, N. B., & Kraft, R. P. 1996, AJ, 111, 1913,
792 doi: [10.1086/117930](https://doi.org/10.1086/117930)
- 793 Valle, G., Dell’Omodarme, M., & Tognelli, E. 2022, A&A,
794 658, A141, doi: [10.1051/0004-6361/202142454](https://doi.org/10.1051/0004-6361/202142454)
- 795 van den Bergh, S. 2010, The Astronomical Journal, 140,
796 1043, doi: [10.1088/0004-6256/140/4/1043](https://doi.org/10.1088/0004-6256/140/4/1043)
- 797 Ventura, P., & D’Antona, F. 2009, A&A, 499, 835,
798 doi: [10.1051/0004-6361/200811139](https://doi.org/10.1051/0004-6361/200811139)
- 799 Ventura, P., D’Antona, F., Mazzitelli, I., & Gratton, R.
800 2001, ApJL, 550, L65, doi: [10.1086/319496](https://doi.org/10.1086/319496)